

PHASE RETRIEVAL VERSES PHASELESS RECONSTRUCTION

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ABSTRACT. In 2006, Balan/Casazza/Edidin [1] introduced the frame theoretic study of phaseless reconstruction. Since then, this has turned into a very active area of research. Over the years, many people have replaced the term *phaseless reconstruction* with *phase retrieval*. Casazza then asked: *Are these really the same?* In this paper, we will show that phase retrieval is equivalent to phaseless reconstruction. We then show, more generally, that phase retrieval by projections is equivalent to phaseless reconstruction by projections. Finally, we study *weak phase retrieval* and discover that it is very different from phaseless reconstruction.

1. INTRODUCTION

Phase retrieval is an old problem in signal processing and has been studied for over 100 years by electrical engineers. Let $x = (a_1, a_2, \dots, a_m)$ and $y = (b_1, b_2, \dots, b_m)$ be vectors in \mathbb{H}_m . We say that x, y have the **same phases** if

$$\text{phase } a_i = \text{phase } b_i, \text{ for all } i = 1, 2, \dots, m.$$

Definition 1.1. Let $\Phi = \{\phi_i\}_{i=1}^n$ be a family of vectors in \mathbb{H}_m (resp. $\{P_i\}_{i=1}^n$ is a family of projections on \mathbb{H}_m) satisfying: for every $x = (a_1, a_2, \dots, a_m)$ and $y = (b_1, b_2, \dots, b_m)$ and

$$(1) \quad |\langle x, \phi_i \rangle| = |\langle y, \phi_i \rangle|, \text{ for all } i = 1, 2, \dots, n.$$

Respectively,

$$(2) \quad \|P_i x\| = \|P_i y\|, \text{ for all } i = 1, 2, \dots, n.$$

(1) If this implies there is a $|\theta| = 1$ so that x and θy have the same phases, we say Φ does **phase retrieval** (Respectively, $\{P_i\}_{i=1}^n$ does **phase retrieval**). Moreover, in the real case, if $\theta = 1$ we say x and y have the **same signs** and if $\theta = -1$ we say x and y have **opposite signs**.

(2) If this implies there is a $|\theta| = 1$ so that $x = \theta y$, we say Φ (does **phaseless reconstruction**). (Respectively, $\{P_i\}_{i=1}^n$ does **phaseless reconstruction**.)

The authors were supported by NSF DMS 1307685; and NSF ATD 1042701 and 1321779; AFOSR DGE51: FA9550-11-1-0245.

In the setting of frame theory, the concept of phaseless reconstruction was introduced in 2006 by Balan/Casazza/Edidin [1]. At that time, they showed that in the real case, a *generic* family of $(2m-1)$ -vectors in \mathbb{R}_m does phaseless reconstruction and no set of $(2m-2)$ -vectors can do this. In the complex case, they showed that a *generic* set of $(4m-2)$ -vectors does phaseless reconstruction. Heinosaari, Mazzarella and Wolf [6] show that n -vectors doing phaseless reconstruction in \mathbb{C}_m requires $n \geq 4m - 4 - 2\alpha$, where α is the number of 1's in the binary expansion of $(m-1)$. Bodmann [3] showed that phaseless reconstruction in \mathbb{C}_m can be done with $(4m-4)$ -vectors. Later, Conca, Edidin, Hering, and Vinzant [5] that a *generic* frame with $(4m-4)$ -vectors does phaseless reconstruction in \mathbb{C}_m . They also show that if $m = 2^k + 1$ then no n -vectors with $n < 4m - 4$ can do phaseless reconstruction. Bandeira, Cahill, Mixon, and Nelson [2] conjectured that for all m , no fewer than $(4m-4)$ -vectors can do phaseless reconstruction. Recently, Vinzant [7] showed that this conjecture does not hold by giving 11 vectors in \mathbb{C}_4 which do phaseless reconstruction.

Over the years, we have started replacing the phrase: *phaseless reconstruction* with the phrase: *phase retrieval*. Casazza at a meeting in 2012 raised the question: *Are these really the same?* In this paper we will answer this question in the affirmative, and the same for phase retrieval by projections, and then show that the notion of *weak phase retrieval* is not equivalent to phaseless reconstruction.

The problem occurred here because of the way we translated the engineering version of *phase retrieval* into the language of frame theory. The engineers are working with the modulus of the Fourier transform and want to recover the phases so they can invert the Fourier transform to discover the signal. So all they need to do is to recover the phase. But in the frame theory version of this, for $x = (a_1, a_2, \dots, a_m)$ we are really trying to recover two things:

- (1) Recover the phases of the a_i .
- (2) Recover $|a_i|$ (which in the engineering case, is already known).

For notation, we will use \mathbb{H}_m to denote a real or complex m -dimensional Hilbert space and for the real and complex cases we use \mathbb{R}_m and \mathbb{C}_m respectively.

2. PHASE RETRIEVAL VERSUS PHASELESS RECONSTRUCTION

We will need the *complement property* from [1].

Definition 2.1. A family of vectors $\{\phi_i\}_{i=1}^n$ in \mathbb{H}_m has the **complement property** if for every $I \subset [n]$, either $\{\phi_i\}_{i \in I}$ spans \mathbb{H}_m or $\{\phi_i\}_{i \in I^c}$ spans \mathbb{H}_m .

We will prove that phase retrieval implies the complement property for both the real and complex cases and even in a more general setting. First, we need to make a couple observations:

Observation 2.2. *If $\{\phi_i\}_{i=1}^n$ does phase retrieval in \mathbb{H}_m , then $\text{span } \{\phi_i\}_{i=1}^n = \mathbb{H}_m$. For otherwise, there would exist $0 \neq x \in \mathbb{H}_m$ so that*

$$\langle x, \phi_i \rangle = \langle 0, \phi_i \rangle = 0, \text{ for all } i = 1, 2, \dots, n,$$

while $x, 0$ do not have the same phase.

Observation 2.3. *If $x = (a_1, a_2, \dots, a_m)$ and $y = (b_1, b_2, \dots, b_m)$ have the same phases, then $a_i = 0$ if and only if $b_i = 0$. I.e. zero has no phase.*

We need a result from [1]. This result was proved in [1] for the real case and it was stated that the same proof works in the complex case. In [2] they state that this claim is *erroneous*, and this proof does not work in the complex case. But, the proof in [1] does work in the complex case and is much easier than the one supplies in [2]. So, we will now show that this argument does in fact work in the coomplex case.

Theorem 2.4 (Balan/Casazza/Edidin). *Let $\Phi = \{\phi_i\}_{i=1}^n$ be vectors in \mathbb{H}_m . If $\Phi = \{\phi_i\}_{i=1}^n$ does phaseless reconstruction, then it has complement property. Moreover, in the real case, these are equivalent while in the complex case they are not equivalent.*

Proof. Assume Φ fails complement property but does phaseless reconstruction. Choose $I \subset [n]$ so that neither of the sets of vectors $\{\phi_i\}_{i \in I}$ or $\{\phi_i\}_{i \in I^c}$ spans \mathbb{H}_m . Choose $\|x\| = 1 = \|y\|$ so that $x \perp \phi_i$ for $i \in I$ and $y \perp \phi_i$ for $i \in I^c$. Then

$$|\langle x + y, \phi_i \rangle| = |\langle x - y, \phi_i \rangle|, \text{ for all } i = 1, 2, \dots, n.$$

Since Φ does phaseless reconstruction, there is a $|\theta| = 1$ so that

$$x + y = \theta(x - y), \text{ and hence } (1 - \theta)x = -(1 + \theta)y.$$

If $\theta = 1$, then $y = 0$ and if $\theta = -1$ then $x = 0$, contradicting the fact that x, y are unit norm. Otherwise,

$$x = \frac{-(1 + \theta)}{1 - \theta}y = dy, \text{ for } d \neq 0.$$

Now,

$$\langle x, \phi_i \rangle = \langle dy, \phi_i \rangle = 0, \text{ for all } i = 1, 2, \dots, n,$$

and hence Φ does not span \mathbb{H}_m contradicting Observation 2.2. \square

Theorem 2.5. *Let $\{P_i\}_{i=1}^n$ be projections onto the subspaces $\{W_i\}_{i=1}^n$ of \mathbb{H}_m which do phase retrieval. Then*

For every orthonormal basis $\{\phi_{i,j}\}_{j=1}^{D_i}$ of W_i , the set $\{\phi_{i,j}\}_{i=1, j=1}^{n, D_i}$ has complement property.

Proof. Suppose $\{W_i\}_{i=1}^n$ does phase retrieval for \mathbb{H}_m , but fails phaseless reconstruction. By Theorem 2.4, there exist an orthonormal basis $\{\phi_{i,j}\}_{j=1}^{D_i}$ of each W_i such that the set $\{\phi_{i,j}\}_{i=1, j=1}^{n, D_i}$ fails the complement property. In other words, there exists $I \subset \{(i, j) : 1 \leq i \leq n \text{ and } 1 \leq j \leq D_i\}$ so that

$\{\phi_{i,j}\}_{(i,j) \in I}$ and $\{\phi_{i,j}\}_{(i,j) \in I^c}$ do not span \mathbb{H}_m . Choose vectors $x, y \in \mathbb{H}_m$ with $\|x\| = 1 = \|y\|$, and $x = (a_1, a_2, \dots, a_m)$ and $y = (b_1, b_2, \dots, b_m)$ such that $x \perp \phi_{i,j}$ for all $(i, j) \in I$ and $y \perp \phi_{i,j}$ for all $(i, j) \in I^c$. Note this choice of vectors forces that for each (i, j) either $\langle x, \phi_{i,j} \rangle = 0$ or $\langle y, \phi_{i,j} \rangle = 0$. Fix $0 \neq c$. Then for each $1 \leq i \leq n$

$$|\langle x + cy, \phi_{i,j} \rangle| = |\langle x - cy, \phi_{i,j} \rangle|, \text{ for all } i, j.$$

Hence,

$$\|P_i(x + cy)\|^2 = \sum_{j=1}^{D_i} |\langle x + cy, \phi_{i,j} \rangle|^2 = \sum_{j=1}^{D_i} |\langle x - cy, \phi_{i,j} \rangle|^2 = \|P_i(x - cy)\|^2.$$

By assumption that $\{P_i\}_{i=1}^n$ does phase retrieval, this implies there is a $|\theta| = 1$ so that $x + cy$ and $\theta(x - cy)$ have the same phases. Assume there exists some $1 \leq i_0 \leq m$ so that $a_{i_0} \neq 0 \neq b_{i_0}$ and let $c = \frac{-a_{i_0}}{b_{i_0}}$. Then

$$(x + cy)_{i_0} = a_{i_0} + cb_{i_0} = a_{i_0} + \frac{-a_{i_0}}{b_{i_0}}b_{i_0} = 0,$$

while

$$(x - cy)_{i_0} = a_{i_0} - \frac{-a_{i_0}}{b_{i_0}} = 2a_{i_0} \neq 0.$$

But this contradicts Observation 2.3. It follows that for every $1 \leq i \leq m$, either $a_i = 0$ or $b_i = 0$. Let $\{e_i\}_{i=1}^m$ be an orthonormal basis for \mathbb{H}_m and let $I = \{1 \leq i \leq m : b_i = 0\}$. Then

$$x + y = \sum_{i \in I} a_i e_i + \sum_{i \in I^c} b_i e_i, \text{ and } x - y = \sum_{i \in I} a_i e_i + \sum_{i \in I^c} (-b_i) e_i.$$

By the above argument, there is a $|\theta| = 1$ so that $x + y$ and $\theta(x - y)$ have this same phase. But this is clearly impossible. This final contradiction completes the proof. \square

We have a number of consequences of Theorem 2.5. Letting the subspaces W_i be one dimensional, this becomes a theorem about vectors.

Corollary 2.6. *If $\Phi = \{\phi_i\}_{i=1}^n$ does phase retrieval in \mathbb{H}_m , then Φ has the complement property. Hence, in the real case, phase retrieval and phaseless reconstruction are equivalent properties.*

We need a result from [4].

Theorem 2.7. *Let $\{P_i\}_{i=1}^n$ be projections onto the subspaces $\{W_i\}_{i=1}^n$ of \mathbb{H}_m . The following are equivalent:*

- (1) $\{P_i\}_{i=1}^n$ does phaseless reconstruction.
- (2) For every orthonormal basis $\{\phi_{i,j}\}_{j=1}^{D_i}$ of W_i , the set $\{\phi_{i,j}\}_{i=1, j=1}^n, D_i$ does phaseless reconstruction.

Combining Theorems 2.5, 2.7:

Corollary 2.8. *In \mathbb{R}_m , a family of projections $\{P_i\}_{i=1}^n$ does phase retrieval if and only if it does phaseless reconstruction.*

In the complex case, the complement property is not equivalent to phaseless reconstruction. We will show that phase retrieval and phaseless reconstruction are equivalent in the complex case in the next section.

3. COMPLEX CASE

Theorem 3.1. [?] *Consider $\Phi = \{\phi_n\}_{n=1}^N \subseteq \mathbb{C}^M$ and the mapping $\mathcal{A} : \mathbb{C}^M/\mathbb{T} \rightarrow \mathbb{R}^N$ defined by $(\mathcal{A}(x))(n) := |\langle x, \phi_n \rangle|^2$. Viewing $\{\phi_n \phi_n^* u\}_{n=1}^N$ as vectors in \mathbb{R}^{2M} , denote $S(u) := \text{span}_{\mathbb{R}}\{\phi_n \phi_n^* u\}_{n=1}^N$. Then the following are equivalent:*

- (a) \mathcal{A} is injective.
- (b) $\dim S(u) \geq 2M - 1$ for every $u \in \mathbb{C}^M \setminus \{0\}$.
- (c) $S(u) = \text{span}_{\mathbb{R}}\{iu\}^\perp$ for every $u \in \mathbb{C}^M \setminus \{0\}$.

For this section we adopt the notation $\langle a, b \rangle_{\mathbb{R}}$ to denote $\text{Re}\langle a, b \rangle$.

Lemma 3.2. *Given $\{\phi_n\}_{n=1}^N \subseteq \mathbb{C}^M$ and any $u \in \mathbb{C}^M$ then $\langle \phi_n \phi_n^* u, iu \rangle_{\mathbb{R}} = 0$*

Proof. The following calculation gives the result almost immediately:

$$\begin{aligned} \langle \phi_n \phi_n^* u, iu \rangle_{\mathbb{R}} &= \langle \langle u, \phi_n \rangle \phi_n, iu \rangle_{\mathbb{R}} = \text{Re}(-i \langle u, \phi_n \rangle \langle \phi_n, u \rangle) \\ &= -\text{Re}(i |\langle u, \phi_n \rangle|^2) = 0. \end{aligned}$$

□

Lemma 3.3. *Given $\{\phi_n\}_{n=1}^N \subseteq \mathbb{C}^M$ and any $u, v \in \mathbb{C}^M$ then for each ϕ_n ,*

$$|\langle u + v, \phi_n \rangle|^2 - |\langle u - v, \phi_n \rangle|^2 = 4 \langle \phi_n \phi_n^* u, v \rangle_{\mathbb{R}}.$$

Proof. Consider the following

$$(3) \quad |\langle u + v, \phi_n \rangle|^2 = |\langle u, \phi_n \rangle|^2 + 2 \text{Re}(\langle u, \phi_n \rangle \overline{\langle v, \phi_n \rangle}) + |\langle v, \phi_n \rangle|^2$$

and

$$(4) \quad |\langle u - v, \phi_n \rangle|^2 = |\langle u, \phi_n \rangle|^2 - 2 \text{Re}(\langle u, \phi_n \rangle \overline{\langle v, \phi_n \rangle}) + |\langle v, \phi_n \rangle|^2.$$

Then subtracting (4) from (3) we obtain

$$|\langle u + v, \phi_n \rangle|^2 - |\langle u - v, \phi_n \rangle|^2 = 4 \text{Re}(\langle u, \phi_n \rangle \overline{\langle v, \phi_n \rangle}) = 4 \langle \phi_n \phi_n^* u, v \rangle_{\mathbb{R}}$$

□

Corollary 3.4. *If $\{\phi_n\}_{n=1}^N$ does phaseless reconstruction and $\langle \phi_n \phi_n^* u, v \rangle_{\mathbb{R}} = 0$ for each n then $u + v = \omega(u - v)$ for $|\omega| = 1$ and thus $v = \frac{2\text{Im}(\omega)}{1+|\omega|^2}u$.*

Proof. If $u + v = \omega u - \omega v$ then $v = \frac{\omega-1}{\omega+1}u = -\frac{(1-\omega)(1+\bar{\omega})}{|1+\omega|^2}u = \frac{2\text{Im}(\omega)}{|1+\omega|^2}u$. □

Lemma 3.5. *Given any u , let $v = \alpha iu$ for $\alpha \in \mathbb{R}$ and let $\omega = \frac{1+\alpha i}{1-\alpha i}$ then $|\omega| = 1$ and $u + v = u(1 + \alpha i) = \frac{1+\alpha i}{1-\alpha i}(u - \alpha iu) = \omega(u - v)$.*

Lemma 3.6. *If $x - y \neq 0$ then $\langle \phi \phi^*(x - y), x + y \rangle_{\mathbb{R}} = 0$.*

Proof. Consider the following calculation,

$$\begin{aligned}\langle \phi\phi^*(x-y), x+y \rangle_{\mathbb{R}} &= \operatorname{Re}((x+y)^*\phi\phi^*(x-y)) \\ &= \operatorname{Re}(|\phi^*x|^2 - x^*\phi\phi^*y + y^*\phi\phi^*x - |\phi y|^2) \\ &= \operatorname{Re}(-x^*\phi\phi^*y + x^*\phi\phi^*y) = 0.\end{aligned}$$

□

Lemma 3.7. *Let $a, b \in \mathbb{C}$ such that $|a| + |b| > 0$. If*

$$\arg(a+b) = \arg(e^{i\theta}(a-b)),$$

then

$$\tan \theta = \frac{2 \operatorname{Im}(\bar{a}b)}{|a|^2 - |b|^2}$$

for $|a| \neq |b|$ and $\theta = \pi/2$ otherwise.

Theorem 3.8. *Phase retrieval implies phaseless construction in the complex case.*

Proof. Suppose $\Phi = \{\phi_n\}_{n=1}^N \subseteq \mathbb{C}^M$ does phase retrieval. Let u, v be non-zero vectors in \mathbb{C}^M such that $\langle \phi_n \phi_n^* u, v \rangle_{\mathbb{R}} = 0$ for all n . Note that Lemma 3.3 ensures that $|\langle u+v, \phi_n \rangle|^2 = |\langle u-v, \phi_n \rangle|^2$ for each n . To apply the results in Theorem ??, we must show $v = \lambda i u$ for some $\lambda \in \mathbb{R}$. For simplicity, denote $u = (u_1, u_2, \dots)$ and $v = (v_1, v_2, \dots)$. Consider the following cases:

Case 1: $u_j v_j = 0$ for all $1 \leq j \leq N$.

Without loss of generality, suppose $u = (e^{i\alpha_1}, 0, \dots)$ and $v = (0, e^{i\beta_2}, \dots)$ for some $\alpha_1, \beta_1 \in \mathbb{R}$. Since Φ does phase retrieval, we have that $u+v$ has the same phase as $e^{i\gamma}(u-v)$, with some real constant γ . In particular $\arg(u_1 + v_1) = \arg(e^{i\gamma}(u_1 - v_1))$, i.e. $\arg(e^{i\alpha_1}) = \arg(e^{i\gamma}e^{i\alpha_1})$. Similarly $\arg(u_2 + v_2) = \arg(e^{i\gamma}(u_2 - v_2))$, i.e. $\arg(e^{i\beta_2}) = \arg(-e^{i\gamma}e^{i\beta_2})$. However the first condition implies $\gamma = 0$ and the second gives $\gamma = \pi$, a contradiction.

Case 1: $u_j v_j \neq 0$ for some $1 \leq j \leq N$.

Without loss of generality, we can assume $u_1 v_1 \neq 0$ and by multiplying by the appropriate constants we may also assume $|u_1| = |v_1| = r_1 > 0$. Then by Lemma 3.7, for each $1 \leq j \leq N$ we have that

$$\tan(\gamma) = \frac{2 \operatorname{Im}(\overline{u_j} v_j)}{|u_j|^2 - |v_j|^2}.$$

By assumption $|u_1| = |v_1|$, therefore $\gamma = \pi/2$ and hence $|u_j| = |v_j|$ for all $1 \leq j \leq N$. So we have shown that

$$u = (r_1 e^{i\alpha_1}, r_2 e^{i\alpha_2}, \dots, r_N e^{i\alpha_N})$$

and

$$v = (r_1 e^{i\beta_1}, r_2 e^{i\beta_2}, \dots, r_N e^{i\beta_N}).$$

Now we claim that $\sin(\beta_j - \alpha_j) = c$ for all j . To see this note that since $\arg(2u_j + v_j) = \arg(e^{i\theta}(2u_j - v_j))$ for all j and fixed θ , then by Lemma 3.7 we see that

$$c = \tan \theta = \frac{4 \operatorname{Im}(\overline{u_j} v_j)}{3r_j^2} = \frac{4}{3} \sin(\beta_j - \alpha_j) \quad \forall 1 \leq j \leq N.$$

For each j , set $a_j = \cos(\beta_j - \alpha_j) = \pm \sqrt{1 - c^2}$. We can express $v = w + ciu$ where

$$w = (a_1 r_1 e^{i\alpha_1}, a_2 r_2 e^{i\alpha_2}, \dots, a_N r_N e^{i\alpha_N}).$$

Now we rewrite

$$v = \left(r_1 e^{i\alpha_1} e^{i(\beta_1 - \alpha_1)}, r_2 e^{i\alpha_2} e^{i(\beta_2 - \alpha_2)}, \dots, r_M e^{i\alpha_M} e^{i(\beta_M - \alpha_M)} \right)$$

and each $e^{i(\beta_j - \alpha_j)} = \cos(\beta_j - \alpha_j) + i \sin(\beta_j - \alpha_j) = a_j + ic$. We must show $w = 0$. Recall that for every n we have

$$0 = \langle \phi_n \phi_n^* u, w + ciu \rangle_{\mathbb{R}} = \langle \phi_n \phi_n^* u, w \rangle_{\mathbb{R}} + \langle \phi_n \phi_n^* u, ciu \rangle_{\mathbb{R}}.$$

By Lemma 3.2 we see that $\langle \phi_n \phi_n^* u, w \rangle_{\mathbb{R}} = 0$ for all n . Note that $w = 0$ if and only if $a_j = 0$ for all j . This is clear since that if $a_1 \neq 0$ then the first component of $a_1 u + w$ is non-zero but the first component of $a_1 u - w$ is 0 (assuming $u_1 \neq 0$) which contradicts to $w = 0$. \square

4. WEAK PHASE RETRIEVAL

We weaken the notion of phase retrieval.

Definition 4.1. Two vectors in \mathbb{H}_m , $x = (a_1, a_2, \dots, a_m)$ and $y = (b_1, b_2, \dots, b_m)$ **weakly have the same phase** if there is a $|\theta| = 1$ so that

$$\operatorname{phase}(a_i) = \theta \operatorname{phase}(b_i), \text{ for all } i = 1, 2, \dots, m, \text{ for which } a_i \neq 0 \neq b_i.$$

In the real case, if $\theta = 1$ we say x, y **weakly have the same signs** and if $\theta = -1$ they **weakly have opposite signs**.

Definition 4.2. A family of vectors $\{\phi_i\}_{i=1}^n$ in \mathbb{H}_m does **weak phase retrieval** if for any $x = (a_1, a_2, \dots, a_m)$ and $y = (b_1, b_2, \dots, b_m)$ in \mathbb{H}_m , with

$$|\langle x, \phi_i \rangle| = |\langle y, \phi_i \rangle|, \text{ for all } i = 1, 2, \dots, m,$$

there is a $|\theta| = 1$ so that

$$\operatorname{phase}(a_i) = \theta \operatorname{phase}(b_i), \text{ for all } i = 1, 2, \dots, m, \text{ for which } a_i \neq 0 \neq b_i.$$

The difference with *phase retrieval* is that we are now allowing $a_i = 0$ and $b_i \neq 0$.

An example of weak phase retrieval which does not yield phase retrieval in \mathbb{R}^m is given by:

Let $\Phi = \{\phi_i\}_{i=1}^{m+1}$ be the column vectors of the matrix:

$$A = \begin{bmatrix} 1 & -1 & 1 \cdots & 1 & 1 \\ 1 & 1 & -1 \cdots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 \cdots & 1 & 1 \end{bmatrix}_{m \times (m+1)}$$

Then for any $x = (a_1, a_2, \dots, a_m)$ and $y = (b_1, b_2, \dots, b_m)$, if

$$|\langle x, \phi_i \rangle|^2 = |\langle y, \phi_i \rangle|^2,$$

then by expanding out and subtracting rows from each other, we will find that:

$$a_i a_j = b_i b_j, \text{ for all } i \neq j.$$

This family of $(m+1)$ -vectors in \mathbb{R}_m does weak phase retrieval. To see this, we need a proposition. Notice that there are too few vectors here to do phaseless reconstruction.

Proposition 4.3. *Let $x = (a_1, a_2, \dots, a_m)$ and $y = (b_1, b_2, \dots, b_m)$ in \mathbb{R}_m . The following are equivalent:*

(1) *We have*

$$\text{sgn}(a_i a_j) = \text{sgn}(b_i b_j), \text{ for all } 1 \leq i \neq j \leq m.$$

(2) *Either x, y have weakly the same signs or they have weakly opposite signs*

Proof. (1) \Rightarrow (2): Let

$$I = \{1 \leq i \leq m : a_i = 0\} \text{ and } J = \{1 \leq i \leq n : b_i = 0\}.$$

Let

$$K = [m] \setminus (I \cup J).$$

So $i \in K$ if and only if $a_i \neq 0 \neq b_i$. Let $i_0 = \min K$. We examine two cases:

Case 1: $\text{sgn } a_{i_0} = \text{sgn } b_{i_0}$.

For any $i_0 \neq k \in K$, $a_{i_0} a_k = b_{i_0} b_k$, implies $\text{sgn } a_k = \text{sgn } b_k$. Since all other coordinates of either x or y are zero, it follows that x, y weakly have the same signs.

Case 2: $\text{sgn } a_{i_0} = -\text{sgn } b_{i_0}$.

For any $i_0 \neq k \in K$, $a_{i_0} a_k = b_{i_0} b_k$ implies $\text{sgn } a_k = -\text{sgn } b_k$. Again, since all other coordinates of either x or y are zero, it follows that x, y weakly have opposite signs.

(2) \Rightarrow (1): This is obvious. □

Theorem 4.4. *Let $x = (a_1, a_2, \dots, a_m)$ and $y = (b_1, b_2, \dots, b_m)$ in \mathbb{R}_m and assume we have:*

$$a_i a_j = b_i b_j, \text{ for all } 1 \leq i \neq j \leq m.$$

Then:

- (1) *Either x, y have weakly the same signs or they have weakly the opposite signs.*
- (2) *One of the following holds:*
 - (i) *There is a $1 \leq i \leq m$ so that $a_i = 0$ and $b_j = 0$ for all $j \neq i$.*
 - (ii) *There is a $1 \leq i \leq m$ so that $b_i = 0$ and $a_j = 0$ for all $j \neq i$.*
 - (iii) *If (i) and (ii) fail and*

$$I = \{1 \leq i \leq m : a_i \neq 0 \neq b_i\},$$

then the following hold:

- (a) *If $i \in I^c$ then $a_i = b_i = 0$.*
- (b) *For all $i \in I$, $|a_i| = |b_i|$.*

Proof. (1) This follows from Proposition 4.3.

(2) (i) Assume $a_i = 0$ but $b_i \neq 0$. Then for all $j \neq i$ we have $a_i a_j = 0 = b_i b_j$ and so $b_j = 0$.

(ii) This is symmetric to (i).

(iii) If (i) and (ii) fail, then by definition, for any i , either both a_i and b_i are zero or they are both non-zero, which proves (a).

Fix $i \in I$. Choose any $j \neq k \in I \setminus \{i\}$. Then

$$a_i a_j = b_i b_j \text{ and } a_i a_k = b_i b_k.$$

Multiplying the left-hand-sides and the right-hand-sides yields,

$$a_i^2 a_j a_k = b_i^2 b_j b_k.$$

Since a_j, a_k, b_j, b_k are all non-zero and $a_j a_k = b_j b_k$, we have that $a_i^2 = b_i^2$. □

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